Forum Geometricorum
Volume 1 (2001) 121–124.

FORUM GEOM
ISSN 1534-1178

A Note on the Feuerbach Point

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Abstract. The circle through the feet of the internal bisectors of a triangle passes through the Feuerbach point, the point of tangency of the incircle and the nine-point circle.

The famous Feuerbach theorem states that the nine-point circle of a triangle is tangent internally to the incircle and externally to each of the excircles. Given triangle ABC, the Feuerbach point F is the point of tangency with the incircle. There exists a family of cevian circumcircles passing through the Feuerbach point. Most remarkable are the cevian circumcircles of the incenter and the Nagel point.[1] In this note we give a geometric proof in the incenter case.

Theorem. *The circle passing through the feet of the internal bisectors of a triangle contains the Feuerbach point of the triangle.*

The proof of the theorem is based on two facts: the triangle whose vertices are the feet of the internal bisectors and the Feuerbach triangle are (a) similar and (b) perspective.

Lemma 1. *In Figure 1, circle $O(R)$ is tangent externally to each of circles $O_1(r_1)$ and $O_2(r_2)$, at A and B respectively. If A_1B_1 is a segment of an external common tangent to the circles (O_1) and (O_2), then*

$$AB = \frac{R}{\sqrt{(R+r_1)(R+r_2)}} \cdot A_1B_1. \tag{1}$$

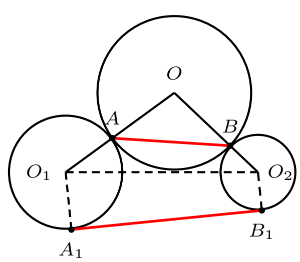

Figure 1

Proof. In the isosceles triangle AOB, $\cos AOB = \frac{2R^2-AB^2}{2R^2} = 1-\frac{AB^2}{2R^2}$. Applying the law of cosines to triangle O_1OO_2, we have

Publication Date: September 4, 2001. Communicating Editor: Paul Yiu.

[1]The cevian feet of the Nagel point are the points of tangency of the excircles with the corresponding sides.

$$O_1O_2^2 = (R+r_1)^2 + (R+r_2)^2 - 2(R+r_1)(R+r_2)\left(1 - \frac{AB^2}{2R^2}\right)$$
$$= (r_1-r_2)^2 + (R+r_1)(R+r_2)\left(\frac{AB}{R}\right)^2.$$

From trapezoid $A_1O_1O_2B_1$, $O_1O_2^2 = (r_1-r_2)^2 + A_1B_1^2$. Comparison now gives A_1B_1 as in (1). □

Consider triangle ABC with side lengths $BC = a$, $CA = b$, $AB = c$, and circumcircle $O(R)$. Let $I_3(r_3)$ be the excircle on the side AB.

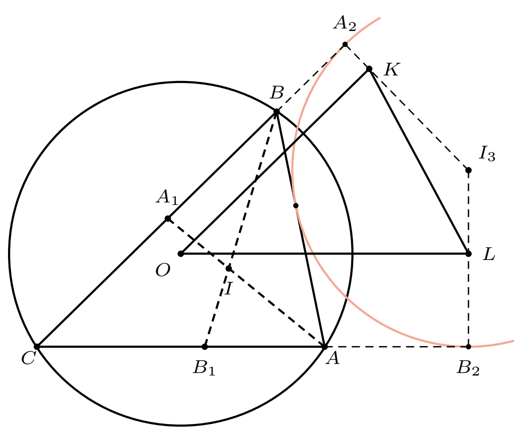

Figure 2

Lemma 2. *If A_1 and B_1 are the feet of the internal bisectors of angles A and B, then*

$$A_1B_1 = \frac{abc\sqrt{R(R+2r_3)}}{(c+a)(b+c)R}. \tag{2}$$

Proof. In Figure 2, let K and L be points on I_3A_2 and I_3B_2 such that $OK//CB$, and $OL//CA$. Since $CA_2 = CB_2 = \frac{a+b+c}{2}$,

$$OL = \frac{a+b+c}{2} - \frac{b}{2} = \frac{c+a}{2}, \qquad OK = \frac{a+b+c}{2} - \frac{a}{2} = \frac{b+c}{2}.$$

Also,

$$CB_1 = \frac{ba}{c+a}, \qquad CA_1 = \frac{ab}{b+c},$$

and

$$\frac{CB_1}{CA_1} = \frac{b+c}{c+a} = \frac{OK}{OL}.$$

Thus, triangle CA_1B_1 is similar to triangle OLK, and

$$\frac{A_1B_1}{LK} = \frac{CB_1}{OK} = \frac{2ab}{(c+a)(b+c)}. \tag{3}$$

Since OI_3 is a diameter of the circle through O, L, K, by the law of sines,

$$LK = OI_3 \cdot \sin LOK = OI_3 \cdot \sin C = OI_3 \cdot \frac{c}{2R}. \tag{4}$$

Combining (3), (4) and Euler's formula $OI_3^2 = R(R+2r_3)$, we obtain (2). □

Now, we prove the main theorem.

(a) Consider the nine-point circle $N(\frac{R}{2})$ tangent to the A- and B-excircles. See Figure 3. The length of the external common tangent of these two excircles is

$$XY = AY + BX - AB = \frac{a+b+c}{2} + \frac{a+b+c}{2} - c = a+b.$$

By Lemma 1,

$$F_1F_2 = \frac{(a+b)\cdot\frac{R}{2}}{\sqrt{(\frac{R}{2}+r_1)(\frac{R}{2}+r_2)}} = \frac{(a+b)R}{\sqrt{(R+2r_1)(R+2r_2)}}.$$

Comparison with (2) gives

$$\frac{A_1B_1}{F_1F_2} = \frac{abc\sqrt{R(R+2r_1)(R+2r_2)(R+2r_3)}}{(a+b)(b+c)(c+a)R^2}.$$

The symmetry of this ratio in a, b, c and the exradii shows that

$$\frac{A_1B_1}{F_1F_2} = \frac{B_1C_1}{F_2F_3} = \frac{C_1A_1}{F_3F_1}.$$

It follows that the triangles $A_1B_1C_1$ and $F_1F_2F_3$ are similar.

(b) We prove that the points F, B_1 and F_2 are collinear. By the Feuerbach theorem, F is the homothetic center of the incircle and the nine-point circle, and F_2 is the internal homothetic center of the nine-point circle and the B- excircle. Note that B_1 is the internal homothetic center of the incircle and the B-excircle. These three homothetic centers divide the side lines of triangle I_2NI in the ratios

$$\frac{NF}{FI} = -\frac{R}{2r}, \qquad \frac{IB_1}{B_1I_2} = \frac{r}{r_2}, \qquad \frac{I_2F_2}{F_2N} = \frac{2r_2}{R}.$$

Since

$$\frac{NF}{FI}\cdot\frac{IB_1}{B_1I_2}\cdot\frac{I_2F_2}{F_2N} = -1,$$

by the Menelaus theorem, F, B_1, and F_2 are collinear. Similarly F, C_1, F_3 are collinear, as are F, A_1, F_1. This shows that triangles $A_1B_1C_1$ and $F_1F_2F_3$ are perspective at F.

From (a) and (b) it follows that

$$\angle C_1FA_1 + \angle C_1B_1A_1 = \angle F_3FF_1 + \angle F_3F_2F_1 = 180^\circ,$$

i.e., the circle $A_1B_1C_1$ contains the Feuerbach point F.

This completes the proof of the theorem.

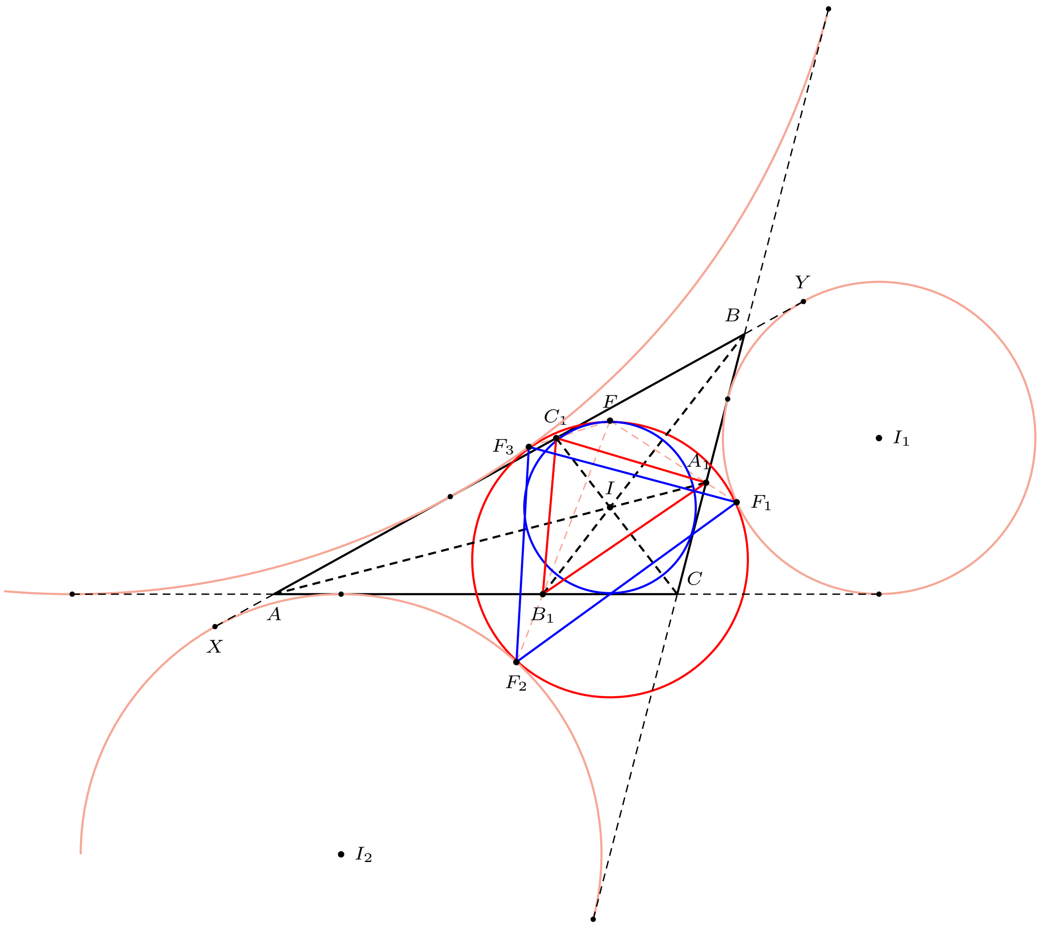

Figure 3

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